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### TABLES FOR THE EXTREME ROOTS OF THE WISHART MATRIX

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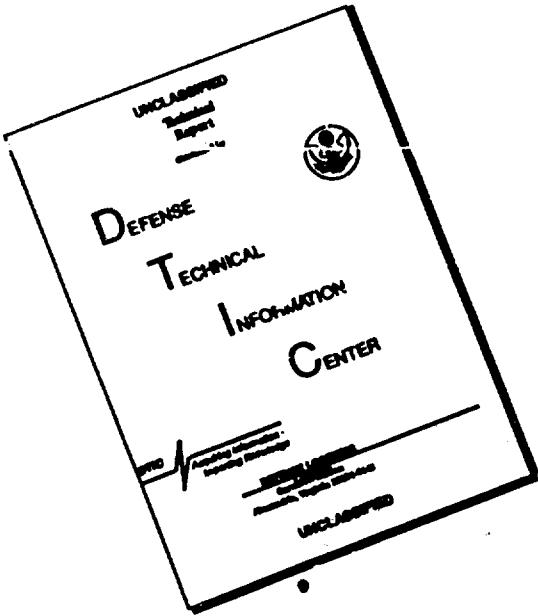
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13. ABSTRACT

Let  $\lambda_1$  and  $\lambda_p$  be respectively the smallest and largest roots of the central  $p \times n$  Wishart matrix with  $n$  degrees of freedom. In this report, the authors gave tables for the exact values of  $U$  for  $p = 2(1)10(2)20$ ,  $\alpha = 0.05, 0.025, 0.01, 0.005$  and different values of  $n$  where

$$P[U^{-1} < \lambda_1 < \lambda_p < U] = (1-2\alpha).$$

Also, exact lower 1%, 2.5%, 5%, 10% points of the distribution of  $\lambda_p$  are given for  $p = 2(1)10(2)20$  and for different values of  $n$ .

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**TABLES FOR THE EXTREME ROOTS  
OF THE WISHART MATRIX**

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AIR FORCE SYSTEMS COMMAND  
UNITED STATES AIR FORCE  
WRIGHT-PATTERSON AIR FORCE BASE, OHIO**

## FOREWORD

This report was prepared for Applied Mathematics Research Laboratory, Aerospace Research Laboratories by D. S. Clemm, P. R. Krishnaiah and V. B. Waikar under Project 7071, "Research in Applied Mathematics". The work of V. B. Waikar was performed at the Aerospace Research Laboratories while in the capacity of an Ohio State University Research Foundation Visiting Research Associate under Contract F 33615-67-C-1758. The present affiliation of V. B. Waikar is Miami University, Oxford, Ohio.

In this report, the authors gave tables for the exact percentage points of the joint distribution of the extreme roots as well as the tables for the exact lower percentage points of the largest root of the Wishart matrix.

The authors wish to thank Miss Eva Brandenburg for typing the manuscript.

## ABSTRACT

Let  $\lambda_1$  and  $\lambda_p$  be respectively the smallest and largest roots of the central  $p \times p$  Wishart matrix with  $n$  degrees of freedom. In this report, the authors gave tables for the exact values of  $U$  for  $p = 2(1)10(2)20$ ,  $\alpha = 0.05$ ,  $0.025$ ,  $0.01$ ,  $0.005$  and different values of  $n$  where

$$P[U^{-1} \leq \lambda_1 \leq \lambda_p \leq U] = (1-2\alpha).$$

Also, exact lower 1%, 2.5%, 5%, 10% points of the distribution of  $\lambda_p$  are given for  $p = 2(1)10(2)20$  and for different values of  $n$ .

## TABLE OF CONTENTS

Section	Page
1. Introduction	1
2. Probability Integral of the Joint Density of the Extreme Roots	1
3. Applications	3
References	6
Tables	

## I. INTRODUCTION

Krishnaiah and Chang [3] gave an expression for the probability integral associated with the joint density of the extreme roots of the Wishart matrix. For computational purposes, this expression is significantly better than other expressions given in the literature ([1], [6]). Using this expression, the authors constructed tables for the percentage points associated with the joint density of the extreme roots of the Wishart matrix by imposing the condition that the lower critical value is equal to the reciprocal of the upper critical value. These tables are useful in the application of Roy's test for testing the hypothesis that the covariance matrix of a multivariate normal population is equal to a specified value. Some computations are also made to check for the accuracy of the approximate tables constructed by Hanumara and Thompson [2] by putting a different restriction on the choice of the critical values. Also tables of the lower percentage points of the largest root of the Wishart matrix are constructed in this paper.

## 2. PROBABILITY INTEGRAL OF THE JOINT DENSITY OF THE EXTREME ROOTS

Let  $X$  be a  $p \times n$  matrix whose columns are independently distributed as multivariate normal with zero mean vector and covariance matrix  $I_p$ , the identity matrix. Also let  $S = XX'$  and let  $\lambda_1 < \lambda_2 < \dots < \lambda_p$  be the latent roots of  $S$ . Then it is well known that  $S$  has Wishart distribution and the joint density of  $\lambda_1, \dots, \lambda_p$  is given by

$$f_1(\ell_1, \dots, \ell_p) = k(p, n) \prod_{i=1}^p [\ell_i^r \exp(-\ell_i/2)] \prod_{i>j}^p (\ell_i - \ell_j) \quad (2.1)$$

$$0 < \ell_1 < \dots < \ell_p < \infty$$

where  $r = (n-p-1)/2$  and

$$k(p, n) = \pi^{p/2} (1/2)^{np/2} \prod_{i=1}^p [r(n+1-i)/2] \Gamma((p+1-i)/2).$$

Krishnaiah and Chang [3] gave an exact expression for the probability integral of the joint density of  $\ell_1$  and  $\ell_p$  which is given by

$$P[L \leq \ell_1 < \ell_p \leq U] = \phi(\psi; p, r, L, U) \quad (2.2)$$

where

$$\phi(\psi; p, r, L, U) = \begin{cases} \Delta(\psi; 2m, r, L, U) & \text{when } p = 2m \\ \sum_{i=0}^{2m} (-1)^i F_{r+i}(U) G_{i+1}(\psi; 2m+1, r, L, U) & \text{when } p = 2m+1 \end{cases} \quad (2.3)$$

$$\text{when } p = 2m+1 \quad (2.4)$$

and

$$\Delta(\psi; 2m, r, L, U) = |(a_{ij})_{i,j=1,\dots,2m}|^{1/2},$$

$$G_t(\psi; 2m+1, r, L, U) = |(a_{ij})_{i,j=1,\dots, t-1, t+1,\dots, 2m+1}|^{1/2} \text{ for } p > 1$$

$$a_{ij} = f_{i+r-1}^{j+r-1}, \quad f_s^t = F_s^t - F_t^s,$$

$$F_s^t = \int_L^U F_s(\theta) \theta^{t-s} \psi(\theta) d\theta, \quad F_s(\theta) = \int_L^\theta \psi(x) x^s dx,$$

$$\psi(x) = \exp(-x/2). \text{ Also, let } G_1(\psi; 1, r, L, U) \equiv 1.$$

In this paper, the authors give tables for the exact values of  $U$  for  $\alpha = 0.05, 0.025, 0.01, 0.005$ ,  $p = 2(1)10(2)20$ ,  $n = (p+1)(1)20(2)30(5)50$  where

$$P[U^{-1} \leq \ell_1 < \ell_p \leq U] = (1-2\alpha). \quad (2.5)$$

The authors also gave lower 1%, 2.5%, 5% and 10% values of the distribution of  $\ell_p$  for  $p = 2(1)10(2)20$ ,  $n = (p+1) (1)20(2)30(5)50$  by using the following known formula [3]:

$$P[\ell_p > U] = 1 - k(p,n) \rho(\psi; p, r, 0, U)$$

Hanumara and Thompson [2] constructed tables for L and U for  $p = 2(1)10$  and various values of r and  $\alpha$  satisfying

$$P[L \leq \ell_1 < \ell_p \leq U] = 1 - 2\alpha \quad (2.6)$$

and

$$P[\ell_1 \geq L] = 1 - \alpha. \quad (2.7)$$

For  $p = 2$ , Thompson [7] computed exact values of L and U. For  $p \geq 3$ , Hanumara and Thompson [2] state that exact values are very difficult to compute and so they computed approximate values  $L_1$  and  $U_1$  of L and U respectively where  $L_1$  and  $U_1$  satisfy

$$P[\ell_1 \geq L_1] = (1 - \alpha) \quad (2.8)$$

$$P[\ell_p \leq U_1] = (1 - \alpha).$$

For  $p = 2$  and 3, they found empirically that  $L_1$  and  $U_1$  are close to L and U respectively. For higher values of  $p (< 10)$  we computed L and U exactly for some typical values of n and  $\alpha$  and found them to be close to  $L_1$  and  $U_1$  given in [2].

### 3. APPLICATIONS

Let  $H_0: \Sigma = \Sigma_0$ ,  $H_a: \Sigma \neq \Sigma_0$  where  $\Sigma$  is the covariance matrix of a p-variate normal distribution and  $\Sigma_0$  is known. Further let S/n be the sample covariance matrix based on a sample of size  $n+1$ . According to the procedure of

Roy [5, p. 30] an acceptance region for testing  $H_0$  against  $H_a$  is given by

$$L \leq \lambda_1 < \lambda_p \leq U \quad (3.1)$$

where  $\lambda_1 < \lambda_2 < \dots < \lambda_p$  are the characteristic roots of  $S\Sigma_0^{-1}$  and  $L < U$  are constants such that

$$P[L \leq \lambda_1 < \lambda_p \leq U | H_0] = 1 - 2\alpha. \quad (3.2)$$

Note that under  $H_0$ ,  $S\Sigma_0^{-1}$  has a Wishart distribution  $W(I, n)$ . Now, the optimum choice of  $L$  and  $U$  (in the sense of maximizing the power) is not known. Thus one simple way of choosing  $L$  and  $U$  is to put  $L = 1/U$  and then use the tables given in this paper.

One can also construct simultaneous confidence intervals for the elements  $\sigma_{ij}$  of the covariance matrix  $\Sigma$  using the tables given here. Roy [5, p. 106] showed that

$$U^{-1} \underline{a}' S \underline{a} \leq \underline{a}' \Sigma \underline{a} \leq L^{-1} \underline{a}' S \underline{a} \quad (3.3)$$

is a set of simultaneous confidence bounds on  $\underline{a}' \Sigma \underline{a}$  for all arbitrary nonnull vectors  $a$  with confidence coefficients  $1 - 2\alpha$  provided  $L$  and  $U$  satisfy equation (2.6). By choosing  $\underline{a}$  appropriately in (3.3) one can obtain simultaneous confidence intervals for  $\sigma_{ij}$ . Thompson [7] derived the following simultaneous confidence intervals for  $\sigma_{ij}$  using Roy's results:

$$\begin{aligned} U^{-1} s_{ii} \leq \sigma_{ii} \leq L^{-1} s_{ii} \quad i = 1, \dots, p \\ |\sigma_{ij} - \frac{1}{2} (U^{-1} + L^{-1}) s_{ij}| \leq \frac{1}{2} (L^{-1} - U^{-1}) (s_{ii} s_{jj})^{1/2} \quad i \neq j. \end{aligned} \quad (3.4)$$

Thus (3.4) represents simultaneous confidence intervals for  $\sigma_{ij}$ ,  $i, j = 1, \dots, p$

with confidence coefficient  $1 - 2\alpha$  where  $L$  and  $U$  satisfy (2.6) and  $S = (s_{ij})$ . Hence putting  $L = 1/U$  and using Table I, one can construct exact simultaneous confidence intervals for  $\sigma_{ij}$  given by (3.4).

As a further application, we mention that these tables are useful in obtaining the simultaneous confidence intervals for the variance components of the two-way layout with unequal variances. Since this application is discussed in detail by Thompson [7,8] and Hanumara and Thompson [2], we omit its discussion here.

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TABLE I\*  
PERCENTAGE POINTS OF THE JOINT DISTRIBUTION  
OF THE EXTREME ROOTS

TABLE 1 (Continued)

P = 2						P = 3					
n	a	.05	.025	.01	.005	n	a	.05	.025	.01	.005
3	12.49	19.38	49.50	99.50	19.63	4	17.70	29.34	74.25	149.25	
4	11.27	13.39	16.50	19.63	18.99	5	15.66	18.06	21.69	25.70	
5	12.58	14.57	17.10	18.99	16.99	6	16.09	19.19	21.94	23.99	
6	14.15	16.22	18.79	20.65	22.45	7	19.64	20.89	23.57	25.65	
7	15.72	17.87	20.55	22.45	24.23	8	20.31	22.62	25.48	27.52	
8	17.26	19.49	22.25	25.95	27.91	9	21.92	24.72	27.25	29.74	
9	18.75	21.07	23.91	25.95	27.63	10	23.59	25.97	28.97	31.12	
10	20.21	22.60	25.54								
11	21.66	24.12	27.13	29.27	30.27	11	25.06	27.58	30.67	32.96	
12	23.09	25.61	28.70	30.89	32.47	12	26.59	29.18	32.33	34.56	
13	24.49	27.06	30.24	32.47	34.05	13	28.09	30.73	33.96	36.25	
14	25.88	29.54	31.76	34.05	34.57	14	29.57	32.29	35.57	37.89	
15	27.25	29.96	33.25	35.59	37.12	15	31.04	33.80	37.14	39.51	
16	28.61	31.39	34.74	37.12	38.63	16	32.50	35.21	38.72	41.13	
17	29.96	32.80	36.20	38.63	40.13	17	33.93	36.80	40.25	42.71	
18	31.31	34.19	37.67	40.13	41.61	18	35.75	38.88	41.80	44.29	
19	32.64	35.57	39.10	41.61	43.08	19	36.77	39.74	42.70	45.93	
20	33.96	36.94	40.53	43.08	45.03	20	38.17	41.18	44.92	47.79	
22	36.57	39.66	43.36	45.98	48.85	22	40.94	44.04	47.78	50.42	
24	39.17	42.35	46.16	48.95	51.69	24	43.68	46.87	50.70	53.41	
26	41.74	45.30	48.92	51.69	54.48	26	46.39	49.67	53.59	56.75	
28	44.27	47.64	51.65	54.48	57.25	28	49.06	52.43	56.44	59.28	
30	46.80	50.24	54.35			30	51.72	55.15	59.26	62.15	
35	53.04	56.59	61.02	64.06		35	58.26	61.89	66.22	69.24	
40	59.19	63.02	67.57	70.75		40	64.71	68.50	73.02	76.19	
45	65.27	69.27	74.02	77.34		45	71.95	75.01	79.71	82.07	
50	71.29	75.46	80.40	87.94		50	77.33	81.44	86.33	89.73	

\*The entries in the table give the values of U where

$$k(p,n) \int_{U^{-1}}^p \cdots \int_{U^{-1}}^p \frac{1}{n} [x_i^r \exp(-x_i^r/2)]_{ij} [x_i - x_j] dx_1 \cdots dx_p = 1 - \alpha.$$

TABLE I (Continued)

TABLE I (Continued)

P = 4		P = 5									
n	$\alpha$	.05	.025	.01	.005	n	$\alpha$	.05	.025	.01	.005
5	22.75	39.01	99.00	199.00		6	27.70	48.74	123.75	248.75	
6	19.94	22.61	26.65	31.63		7	24.17	27.05	31.49	37.49	
7	21.27	23.63	26.60	28.79		8	25.49	28.01	31.14	33.45	
8	22.98	25.39	28.33	30.43		9	27.23	29.76	32.97	35.09	
9	24.70	27.16	30.17	32.32		10	28.98	31.58	34.74	36.98	
10	25.36	28.90	31.98	34.18							
11	28.01	30.60	33.76	36.00		11	30.70	33.36	36.58	39.95	
12	29.63	32.28	35.50	37.77		12	32.39	35.10	38.38	40.70	
13	31.21	33.92	37.19	39.52		13	34.05	36.80	40.15	42.52	
14	32.77	35.54	38.88	41.25		14	35.67	38.50	41.90	44.31	
15	34.32	37.13	40.53	42.94		15	37.29	40.15	43.61	46.06	
16	35.85	38.71	42.16	44.60		16	38.89	41.70	45.29	47.79	
17	37.35	40.27	43.78	46.25		17	40.44	43.41	46.97	49.49	
18	39.84	41.91	45.36	47.89		18	42.00	45.00	48.62	51.19	
19	40.33	43.33	46.95	49.51		19	43.54	46.59	50.24	52.83	
20	41.79	44.84	48.51	51.10		20	45.06	48.15	51.87	54.47	
22	44.68	47.83	51.59	54.25		22	48.07	51.24	55.95	57.73	
24	47.55	50.77	54.63	57.36		24	51.02	54.28	58.18	60.93	
26	50.36	53.67	57.62	60.41		26	53.95	57.27	61.65	64.97	
28	53.16	56.54	60.57	63.42		28	56.93	60.23	64.30	67.18	
30	55.92	59.36	63.49	66.40		30	59.68	63.16	67.31	70.24	
35	62.72	66.34	70.67	73.71		35	66.70	70.34	74.69	77.75	
40	69.39	73.18	77.69	80.86		40	73.57	77.37	81.91	95.07	
45	75.95	79.90	84.60	87.89		45	80.37	84.28	88.98	92.25	
50	82.43	86.53	91.40	94.79		50	86.99	91.09	95.95	99.34	

TABLE I (Continued)

TABLE I (Continued)

P = 6		P = 7									
n	$\alpha$	.05	.025	.01	.005	n	$\alpha$	.05	.025	.01	.005
7	32.59	58.49	148.49	298.50		8	37.40	68.24	173.24	749.25	
8	28.36	31.43	36.24	43.37		9	32.54	35.77	40.92	49.29	
9	29.67	32.32	35.61	38.04		10	33.92	36.60	40.02	42.56	
10	31.44	34.09	37.73	39.62							
11	33.21	35.92	39.21	41.54		11	35.61	38.38	41.74	44.12	
12	34.97	37.74	41.07	43.44		12	37.40	40.23	43.64	46.04	
13	36.69	39.51	42.91	45.32		13	39.19	42.05	45.52	47.06	
14	38.38	41.25	44.71	47.15		14	40.94	43.85	47.36	49.94	
15	40.05	42.96	46.47	48.96		15	42.65	45.62	49.18	51.70	
16	41.70	44.66	48.21	50.73		16	44.36	47.36	50.98	53.53	
17	43.31	46.33	49.94	52.50		17	46.03	49.08	52.75	55.33	
18	44.92	47.97	51.63	54.23		18	47.68	50.78	54.48	57.11	
19	46.52	49.51	53.31	55.94		19	49.32	52.45	56.21	58.95	
20	48.08	51.22	54.98	57.64		20	50.94	54.11	57.92	60.60	
22	51.19	54.41	58.25	60.97		22	54.13	57.38	61.27	64.02	
24	54.24	57.54	61.47	64.25		24	57.26	60.60	64.57	67.39	
26	57.25	60.52	64.64	67.48		26	60.35	63.76	67.82	70.59	
28	60.22	63.66	67.76	70.66		28	63.40	66.81	71.02	73.92	
30	63.16	66.67	70.85	73.79		30	66.42	69.95	74.17	77.13	
35	70.37	74.13	78.41	81.48		35	73.81	77.49	81.90	94.09	
40	77.43	81.24	85.79	88.98		40	81.02	84.87	89.44	92.64	
45	84.35	88.31	93.03	96.33		45	88.11	92.00	96.82	100.13	
50	91.17	95.27	100.15	103.55		50	95.08	99.19	104.08	117.69	

TABLE I (Continued)

TABLE I (Continued)

P = .8						P = .9					
n	.05	.025	.01	.005	.001	n	.05	.025	.01	.005	n
9	42.19	77.98	197.99	398.00		10	46.95	87.73	222.74	447.75	
10	36.69	40.06	45.54	55.28		11	40.63	44.34	50.12	51.35	
11	37.96	40.84	44.39	47.04		12	42.08	45.06	48.74	51.48	
12	39.75	42.63	46.09	48.56		13	43.98	46.85	50.43	52.97	
13	41.57	44.49	48.01	50.49		14	45.72	48.73	52.34	54.91	
14	43.38	46.34	49.91	52.43		15	47.54	50.59	54.25	56.94	
15	45.14	48.15	51.77	54.33		16	49.32	52.42	56.17	58.75	
16	46.89	49.94	53.60	56.20		17	51.09	54.23	57.99	60.64	
17	48.61	51.71	55.41	58.04		18	52.83	56.01	59.82	62.50	
18	50.30	53.45	57.20	59.85		19	54.55	57.77	61.61	64.74	
19	51.99	55.16	58.97	61.66		20	56.26	59.51	63.40	66.15	
20	53.65	56.87	60.71	63.43		22	59.60	62.93	66.91	69.72	
22	56.92	60.21	64.15	66.92		24	62.91	66.29	70.35	73.21	
24	60.14	63.51	67.52	70.36		26	66.13	69.60	73.73	76.65	
25	63.31	66.74	70.64	73.73		28	69.32	72.85	77.07	80.02	
26	66.42	69.93	74.11	77.04		30	72.47	76.06	80.34	83.76	
30	69.51	73.07	77.32	80.31		35	80.17	83.92	88.38	91.51	
35	77.06	80.78	85.21	88.32		40	87.69	91.57	96.20	99.44	
40	84.43	88.29	92.89	96.11		45	95.05	99.07	103.84	107.19	
45	91.66	95.66	100.42	103.74		50	102.28	106.42	111.35	114.79	
50	98.76	102.89	107.80	111.22							

TABLE I (Continued)

TABLE I (Continued)

		P=1.0				P=1.2				
		.05	.025	.01	.005		.05	.025	.01	.005
n						n				
11	51.68	97.48	247.49	497.50	497.50	13	61.07	116.97	296.99	595.99
12	44.96	48.50	54.66	57.50	55.87	14	53.19	57.07	63.66	79.99
13	46.19	49.26	53.05	57.35	57.4	15	54.38	57.63	61.63	64.62
14	48.00	51.05	54.74	56.6	59.29	16	56.20	59.43	63.29	66.03
15	49.95	52.94	58.57	61.24	61.24	17	59.07	51.73	65.22	67.97
16	51.68	54.82	60.47	63.16	63.16	18	59.93	63.21	67.15	69.93
17	53.48	56.66	62.33	65.05	65.05	19	61.76	65.08	69.07	71.97
18	55.27	58.49	64.17	66.92	66.92	20	67.57	70.95	73.79	
19	57.03	60.28	66.00	68.76	68.76					
20	58.76	62.05								
22	62.19	65.56	69.57	72.41	72.41	22	67.13	70.56	74.65	77.55
24	65.55	68.98	73.08	75.96	75.96	24	70.61	74.11	78.28	81.22
26	68.85	72.34	76.52	79.46	79.46	26	74.04	77.60	81.85	84.84
28	72.10	75.67	79.91	82.90	82.90	28	77.41	81.03	85.35	88.39
30	75.31	78.94	83.26	86.29	86.29	30	80.77	84.41	88.80	91.88
35	83.16	86.93	91.43	94.58	94.58	35	88.85	92.67	97.23	100.42
40	90.81	94.72	99.38	102.63	102.63	40	96.75	100.71	105.42	108.71
45	98.30	102.34	107.15	110.50	110.50	45	104.48	108.56	113.42	116.91
50	105.66	109.82	114.77	118.21	118.21	50	112.05	116.25	121.25	124.73

TABLE I (Continued)

TABLE I (Continued)

<b>P=14</b>						<b>P=16</b>					
<b>n</b>	<b>a</b>	<b>.05</b>	<b>.025</b>	<b>.01</b>	<b>.005</b>	<b>n</b>	<b>a</b>	<b>.05</b>	<b>.025</b>	<b>.01</b>	<b>.005</b>
15	70.41	136.47	346.49	696.49		17	79.71	155.97	395.99	795.99	
16	61.40	65.49	72.59	92.65		18	69.59	73.87	81.42	105.38	
17	62.55	65.95	70.15	73.27		19	70.70	74.24	78.51	81.87	
18	64.37	67.75	71.79	74.65		20	72.53	76.04	80.23	83.20	
19	66.26	69.57	73.72	76.59							
20	68.13	71.57	75.67	78.56							
22	71.92	75.31	79.49	82.43		22	76.31	79.88	84.13	87.12	
24	75.42	78.98	83.24	96.22		24	80.03	83.66	87.98	91.01	
25	78.96	92.58	86.91	89.94		26	83.67	87.35	91.75	94.83	
28	92.44	96.12	90.51	93.60		28	87.25	90.99	95.46	98.58	
30	85.87	89.60	94.07	97.19		30	90.77	94.57	99.10	102.26	
35	94.23	98.11	102.72	105.96		35	99.37	103.30	107.98	111.24	
40	102.36	106.37	111.14	114.47		40	107.72	111.77	116.60	119.96	
45	110.31	114.43	119.34	122.76		45	115.86	120.03	124.99	128.44	
50	119.09	122.32	127.37	130.88		50	123.83	128.11	133.20	136.75	

TABLE I (Continued)

TABLE I (Continued)

		P=18						P=20					
		.05	.025	.01	.005	n	a	.05	.025	.01	.005		
19	88.98	175.46	445.48	895.49								98.94	170.93
20	77.75	82.22	90.20	118.15								97.03	100.19
22	90.66	94.30	98.64	91.71								100.94	104.10
24	84.47	88.16	92.55	95.63	22	85.90	90.55					104.82	108.03
26	88.21	91.96	96.42	99.54	24	88.79	92.54					108.54	111.99
28	91.89	95.68	100.21	103.38	26	92.61	96.41					100.23	104.10
30	95.51	99.35	103.94	107.15	28	95.38	100.23					103.99	108.54
					30	100.08	103.99						
35	104.32	108.29	113.03	116.35	35	109.09	113.13					117.92	121.26
40	112.86	116.97	121.84	125.24	40	117.87	121.98					126.91	130.75
45	121.19	125.40	130.42	133.91	45	126.34	130.60					135.66	139.19
50	129.34	133.66	138.80	142.37	50	134.65	139.01					144.19	147.80

TABLE II\*\*  
LOWER PERCENTAGE POINTS OF THE  
LARGEST ROOT

TABLE III (Continued)

n	P = 2					P = 3				
	a	.01	.025	.05	.10	a	.01	.025	.05	.10
3	.69	.99	1.32	1.78	2.11	4	2.71	2.87	3.42	4.15
4	1.25	1.66	2.08	2.69	3.59	5	3.18	3.84	4.48	5.72
5	1.86	2.36	2.89	3.51	4.51	6	4.07	4.83	5.54	6.47
6	2.52	3.12	3.72	4.57	5.45	7	4.99	5.81	6.60	7.51
7	3.21	3.90	4.57	5.43	6.39	8	5.90	6.80	7.65	8.73
8	3.93	4.69	5.50	6.30	7.34	9	6.82	7.79	8.70	9.85
9	4.67	5.42	6.31	7.18	8.29	10	7.75	8.79	9.75	10.95
10	5.42									
11	6.18	7.14	8.07	9.24	11	8.69	9.78	10.80	12.07	
12	6.96	7.99	8.97	10.20	12	9.64	10.78	11.94	13.17	
13	7.74	8.94	9.87	11.16	13	10.58	11.77	12.88	14.27	
14	8.53	9.69	10.77	12.12	14	11.52	12.77	13.92	15.76	
15	9.33	10.54	11.67	13.04	15	12.47	13.76	14.96	16.46	
16	10.15	11.40	12.58	14.05	16	13.41	14.76	16.00	17.54	
17	10.97	12.27	13.49	15.01	17	14.35	15.76	17.04	18.53	
18	11.79	13.14	14.41	15.98	18	15.31	16.76	18.08	19.71	
19	12.62	14.02	15.32	16.35	19	16.27	17.75	19.11	21.79	
20	13.44	14.90	16.24	17.92	20	17.23	18.75	20.15	21.97	
22	15.13	16.57	18.09	19.85	22	19.14	20.75	22.21	24.72	
24	16.92	18.45	19.95	21.80	24	21.06	22.74	24.28	25.17	
26	18.53	20.23	21.81	23.74	26	22.98	24.74	26.34	28.79	
28	20.24	22.03	23.68	25.69	28	24.91	26.74	28.40	30.44	
30	21.97	23.84	25.55	27.64	30	26.94	28.74	30.46	32.57	
35	26.32	28.37	30.24	32.52	35	31.67	33.73	35.60	37.97	
40	30.73	32.95	34.96	37.41	40	36.52	38.73	40.73	42.15	
45	35.16	37.55	39.70	42.30	45	41.37	43.73	45.85	49.42	
50	39.64	42.16	44.45	47.20	50	46.23	48.72	50.96	53.57	

\*The entries in the table are the values of U where

$$P[\ell_p \geq U] = 1 - k(p, n) \sum_{i=1}^r \dots \sum_{i>j}^r [\ell_i^r \exp(-\ell_i^r/2)] \prod_{i>j}^r (\ell_i - \ell_j) d\ell_1 \dots d\ell_p = 1 - \alpha$$

$$0 < \ell_1 < \dots < \ell_p < U$$

TABLE II (Continued)

TABLE II (Continued)

n	P = 4					P = 5				
	$\alpha$	.01	.025	.05	.10	n	$\alpha$	.01	.025	.05
5	4.47	5.23	5.98	6.93	6.93	6	6.98	7.91	8.80	9.91
6	5.56	6.40	7.21	8.24	8.24	7	9.19	9.19	10.14	11.33
7	6.64	7.56	8.43	9.53	9.53	8	9.39	10.47	11.47	12.73
8	7.71	8.70	9.63	10.81	10.81	9	10.58	11.72	12.78	14.09
9	8.78	9.83	10.82	12.05	12.05	10	11.76	12.96	14.06	15.43
10	9.85	10.96	12.00	13.29	13.29					
11	10.91	12.07	13.16	14.52	14.52	11	12.93	14.18	15.33	16.75
12	11.97	13.19	14.32	15.74	15.74	12	14.19	15.39	16.59	18.07
13	13.02	14.30	15.46	16.94	16.94	13	15.26	16.60	17.84	19.36
14	14.08	15.40	16.63	18.14	18.14	14	16.41	17.90	19.07	20.65
15	15.13	16.51	17.77	19.33	19.33	15	17.56	18.99	20.30	21.92
16	16.19	17.61	18.91	20.51	20.51	16	18.70	20.17	21.52	23.18
17	17.24	18.71	20.04	21.69	21.69	17	19.83	21.35	22.74	24.44
18	18.29	19.80	21.17	22.87	22.87	18	20.97	22.52	23.95	25.69
19	19.34	20.89	22.30	24.03	24.03	19	22.10	23.69	25.14	26.93
20	20.39	21.98	23.42	25.20	25.20	20	23.27	24.86	26.34	28.15
22	22.49	24.14	25.66	27.51	27.51	22	25.46	27.16	28.72	30.51
24	24.59	26.31	27.89	29.81	29.81	24	27.59	29.47	31.08	33.94
26	26.67	28.46	30.10	32.10	32.10	26	29.91	31.76	33.42	35.45
28	28.75	30.61	32.31	34.37	34.37	28	32.12	34.03	35.75	37.85
30	30.83	32.76	34.51	36.64	36.64	30	34.33	36.29	38.07	40.23
35	38.02	38.10	39.96	42.27	42.27	35	39.91	41.92	43.83	46.13
40	41.19	43.42	45.42	47.85	47.85	40	45.26	47.50	49.52	51.07
45	46.36	48.72	50.84	53.40	53.40	45	50.69	53.05	55.18	57.75
50	51.52	54.00	56.23	58.92	58.92	50	56.09	58.57	60.80	63.49

TABLE II (Continued)

TABLE II (Continued)

		P= 6				P= 7				
		.01	.025	.05	.10	n	.01	.025	.05	.10
7	9.69	10.76	11.78	13.04						
8	11.00	12.15	13.21	14.54	8	12.57	13.76	14.89	16.27	
9	12.30	13.51	14.62	16.00	9	13.96	15.22	16.38	17.83	
10	13.53	14.85	16.00	17.44	10	15.32	16.65	17.85	19.74	
11	14.85	16.15	17.36	18.85	11	16.67	18.04	19.29	20.94	
12	16.10	17.46	18.71	20.24	12	18.91	19.42	20.72	22.70	
13	17.35	18.75	20.03	21.62	13	19.34	20.79	22.11	23.75	
14	18.59	20.02	21.35	22.98	14	20.64	22.13	23.50	25.19	
15	19.81	21.29	22.65	24.32	15	21.93	23.47	24.87	26.59	
16	21.03	22.55	23.95	25.66	16	23.21	24.79	26.23	27.99	
17	22.23	23.90	25.23	26.98	17	24.49	26.10	27.57	29.37	
18	23.43	25.04	26.50	28.29	18	25.76	27.40	28.91	30.75	
19	24.63	26.27	27.77	29.60	19	27.01	28.70	30.23	32.11	
20	25.82	27.50	29.03	30.90	20	28.25	29.99	31.55	33.46	
22	28.20	29.94	31.53	33.46	22	30.75	32.54	34.16	36.13	
24	30.55	32.36	34.00	36.01	24	33.21	35.06	36.74	38.78	
26	32.89	34.76	36.46	38.53	26	35.55	37.56	39.29	41.40	
28	35.21	37.14	38.90	41.03	28	38.98	40.05	41.83	43.99	
30	37.52	39.52	41.32	43.51	30	40.48	42.51	44.34	46.55	
35	43.26	45.39	47.31	49.65	35	46.45	48.61	50.56	52.91	
40	48.94	51.20	53.24	55.70	40	52.34	54.63	56.68	59.16	
45	54.59	56.97	59.11	61.69	45	58.19	60.58	62.74	65.74	
50	60.19	62.69	64.93	67.63	50	63.98	66.48	68.74	71.45	

TABLE II (Continued)

TABLE II (Continued)

P = .8						P = .9					
n	$\alpha$	.01	.025	.05	.10	n	$\alpha$	.01	.025	.05	.10
9	15.56	16.88	18.09	19.59	21.19	10	19.64	20.06	21.36	22.95	22.95
10	17.01	18.38	19.64	21.19							
11	18.44	19.86	21.15	22.75	24.28	11	20.15	21.61	22.95	24.59	24.59
12	19.85	21.30	22.64	24.11	25.79	12	21.62	23.12	24.50	26.19	26.19
13	21.23	22.74	24.11	25.56	27.29	13	23.07	24.62	26.03	27.75	27.75
14	22.60	24.15	25.56	27.29	28.76	14	24.51	26.09	27.54	29.71	29.71
15	23.96	25.55	26.99	28.40	30.21	15	25.92	27.55	29.03	30.84	30.84
16	25.31	26.93	28.40	29.81	31.65	16	27.32	28.99	30.50	32.74	32.74
17	26.64	28.30	29.81	31.20	33.08	17	28.71	30.41	31.96	33.94	33.94
18	27.97	29.66	31.20	32.58	34.49	18	30.09	31.92	33.39	35.71	35.71
19	29.28	31.01	32.58	33.95	35.89	19	31.46	33.22	34.83	36.79	36.79
20	30.59	32.34	33.95			20	32.81	34.61	36.24	38.23	38.23
22	33.17	35.00	36.66	38.67		22	35.49	37.35	39.04	41.59	41.59
24	35.73	37.62	39.33	41.41	44.11	24	38.14	40.07	41.81	43.92	43.92
26	38.27	40.21	41.98	44.11	46.60	26	40.77	42.75	44.54	46.71	46.71
28	40.79	42.79	44.60	46.79	49.45	28	43.36	45.40	47.24	49.46	49.46
30	43.29	45.34	47.19			30	45.94	48.03	49.92	52.23	52.23
35	49.45	51.64	53.61	55.99		35	52.31	54.52	56.51	58.92	58.92
40	55.54	57.94	59.92	52.42		40	58.57	50.90	52.99	55.52	55.52
45	61.56	63.97	66.14	68.77		45	64.75	67.19	69.38	72.82	72.82
50	67.52	70.04	72.30	75.03		50	70.97	73.41	75.69	79.43	79.43

TABLE II (Continued)

TABLE II (Continued)

P=10

P=12

n	$\alpha$	.01	.025	.05	.10	n	$\alpha$	.01	.025	.05	.10
11	21.81	23.31	24.70	26.38		13	28.30	29.97	31.49	33.75	
12	23.34	24.90	26.31	28.04		14	29.67	31.61	33.16	35.05	
13	24.86	26.45	27.90	29.67		15	31.27	31.47	33.21	34.80	
14	26.34	27.98	29.46	31.27		16	32.05	32.52	34.00	36.41	
15	27.82	29.49	31.00	32.41		17	34.41	34.95	36.37	38.01	
16	29.27	30.98	32.52	34.41		18	35.95	37.47	37.92	39.59	
17	30.72	32.45	34.03	35.52		19	37.47	37.57	39.45	41.14	
18	32.14	33.91	35.52	37.47		20	38.98	39.98	40.97	42.69	
19	33.55	35.36	36.99	38.98							
20	34.96	36.79	38.45	40.48							
22	37.73	39.63	41.34	43.43		24	42.00	43.96	45.74	47.99	
24	40.46	42.42	44.19	46.33		26	44.89	46.91	48.73	50.94	
26	43.17	45.18	47.00	49.20		28	47.75	49.82	51.69	53.95	
28	45.65	47.91	49.78	52.03		30	50.56	52.69	54.60	56.92	
30	48.50	50.62	52.52	54.84							
35	55.03	57.27	59.29	61.73		35	60.21	62.50	64.56	67.05	
40	61.46	63.82	65.93	69.49		40	66.94	69.34	71.49	74.09	
45	67.80	70.26	72.46	75.12		45	77.56	76.06	78.30	81.39	
50	74.06	76.62	78.92	81.68		50	80.08	82.68	85.01	87.91	

TABLE II (Continued)

TABLE II (Continued)

		P=14						P=16					
		.01	.025	.05	.10			.01	.025	.05	.10		
n	a	34.97	36.79	38.43	40.43								
15	35.62	38.46	40.13	42.15									
16	38.24	40.11	41.81	43.87	17	41.78	43.72	45.47	47.50				
17	39.93	41.74	43.47	45.56	18	43.45	45.42	47.20	49.36				
18	41.42	43.35	45.10	47.23	19	45.10	47.10	48.91	51.13				
19	42.91	44.95	46.72	48.88	20	46.75	48.77	50.60	52.81				
n	a	46.06	48.08	49.91	52.12	22	49.97	52.04	53.92	56.19			
22	49.09	51.17	53.05	55.31	24	53.13	55.26	57.19	59.53				
24	52.08	54.21	56.13	58.45	26	56.24	58.42	60.39	62.75				
26	55.03	57.21	59.17	61.54	28	59.31	61.54	63.54	65.37				
28	57.95	60.17	62.17	64.59	30	62.33	64.61	66.66	69.13				
n	a	65.10	67.44	69.54	72.07	35	69.76	72.14	74.28	76.86			
35	72.09	74.54	76.77	79.37	40	77.00	79.49	81.72	84.40				
40	79.97	81.51	83.79	86.53	45	84.10	86.59	89.00	91.78				
45	85.73	88.37	90.73	93.56	50	91.09	93.76	96.15	99.03				

TABLE II (Continued)

TABLE II (Continued)

P=18						P=20					
n	a	.01	.025	.05	.10	n	a	.01	.025	.05	.10
19	48.69	50.75	52.60	54.84		22	57.41	59.60	61.56	62.94	
20	50.39	52.47	54.35	56.62		24	60.80	63.04	65.05	67.43	
22	53.74	55.88	57.80	60.12		24	60.80	63.04	65.05	67.43	
24	57.02	59.21	61.17	63.55		26	64.14	66.42	68.47	70.95	
25	60.25	62.48	64.49	66.92		26	64.14	66.42	68.47	70.95	
28	63.43	65.71	67.76	70.23		28	67.42	69.75	71.84	74.76	
30	66.56	68.89	70.97	73.49		30	70.66	73.93	75.15	77.72	
35	74.23	76.66	78.84	81.47		35	78.56	81.03	83.25	85.92	
40	81.71	84.24	86.50	89.23		40	86.25	88.83	91.12	93.89	
45	89.13	91.65	94.00	96.82		45	93.78	96.44	98.82	101.63	
50	96.22	98.93	101.35	104.26		50	101.16	103.90	106.36	109.71	